Topics

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Mixed Equilibrium in a Pure Location Game: The Case of $n \geq 4$ Firms

Abstract: The Hotelling game of pure location allows interpretations in spatial competition, political theory, and strategic forecasting. In this paper, the doubly symmetric mixed-strategy equilibrium for $n \geq 4$ firms is characterized as the solution of a well-behaved boundary value problem. The analysis suggests that, in contrast to the cases $n = 3$ and $n \to \infty$, the equilibrium for a finite number of $n \geq 4$ firms tends to overrepresent locations at the periphery of its support interval. Moreover, in the class of examples considered, an increase in the number of firms universally leads to a wider range of location choices and to a more dispersed distribution of individual locations.

Keywords: Hotelling game, mixed-strategy equilibrium, shooting method, strategic forecasting

JEL Classification: C72, D43, D72, D82, L13

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1 Introduction

The present paper deals with what is known as the Hotelling (1929) model of pure location, in which each of a given finite number of firms simultaneously and independently chooses a location on the unit interval so as to maximize its expected market share.[1] While traditional applications related to spatial competition and political theory remain important, the framework has more recently

1 The pure location model is a simplified variant of Hotelling's original set-up (cf. Chamberlin 1938, Appendix C). For an introduction to the literature on spatial competition, see Gabszewicz and Thisse (1992).

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been recognized as capturing also strategic aspects of the competition between professional forecasters.[2]

The game-theoretic analysis of the pure location game was initially concerned mainly with equilibria in pure strategies (Lerner and Singer 1937; Eaton and Lipsey 1975).[3] However, for $n = 3$, there is no pure-strategy equilibrium. Moreover, in cases where the pure-strategy equilibrium exists, the equilibrium typically vanishes if the density function associated with the underlying distribution of consumer preferences is either strictly convex or strictly concave (Osborne and Pitchik 1986). Finally, pure-strategy equilibria may sometimes be harder to coordinate upon (see, e.g., Xefteris 2014).[4] It should, therefore, not come as a surprise that attention has also been devoted to the analysis of mixed-strategy equilibria.

Of particular interest has been the so-called doubly symmetric equilibrium in which each firm uses the same mixed strategy and in which, in addition, the distribution of individual choices that represents the mixed equilibrium strategy is symmetric with respect to the midpoint of the location interval. Shaked (1982) showed that the doubly symmetric mixed-strategy equilibrium with $n = 3$ firms is unique, and uniform on the interval $[1/4, 3/4]$. For general $n \geq 3$, Osborne and Pitchik (1986) proved that there exists an atomless doubly symmetric mixed-strategy equilibrium, where the support is necessarily an interval if consumer preferences are distributed uniformly. Moreover, as the number of firms n goes to infinity, any convergent sequence of twice continuously differentiable equilibrium distributions must ultimately approach the underlying distribution of customer preferences. Despite these general insights, however, a more qualitative description of the mixed-strategy equilibrium for $n \geq 4$ firms remained elusive.[5]

The contribution of this paper is a re-formulation of the equilibrium condition for the location game with $n \geq 4$ firms in terms of a well-behaved boundary value problem. Based on the resulting characterization of the equilibrium distribution, a numerical solution is obtained for small values of n by studying

2 See Laster, Bennett, and Geoum (1999), Ottaviani and Sørensen (2006), and Marinovic, Ottaviani, and Sørensen (2013).

3 See also Graitson (1982), Denzau, Kats, and Slutsky (1985), and Cox (1987).

4 As an illustration, consider the location model with $n = 5$ firms. In the pure-strategy equilibrium, two firms locate at the first sextile, two others at the fifth sextile, and one firm at the market center. Thus, the market share of the central firm is twice as large as that of its competitors, making coordination on the pure-strategy equilibrium potentially difficult.

5 Osborne and Pitchik (1986, p. 227) write: "Even if C is uniform this is a difficult problem – (2) is a nonlinear integral equation, about which little in general is known." Also the brute-force approach via discretization of the strategy space has remained ineffective. See, e.g. Huck, Müller, and Vriend (2002) for the case of $n = 4$ firms.

trajectories that depart from the midpoint of the location interval. It turns out that, in all cases considered, the doubly symmetric equilibrium involves a tendency to overrepresent locations at the periphery of its support interval. Moreover, an increase in the number of firms universally leads to a wider range of locations that are used in equilibrium and to a more dispersed distribution of individual choices.[6]

The remainder of the paper is structured as follows. Section 2 reviews the location game. Section 3 discusses the first-order condition. The equilibrium is characterized and discussed in Section 4. Section 5 outlines the numerical analysis. Section 6 concludes. An Appendix contains technical proofs.

2 Review of the location game

This section introduces the set-up and reviews some well-known results regarding the doubly symmetric mixed-strategy equilibrium of the location game. To avoid confusion, the framework will be presented primarily in terms of the original interpretation, i.e. in terms of firms choosing locations. Alternative interpretations will be allowed again in the concluding section.

In the location game, a finite number of $n \geq 3$ firms independently and simultaneously choose a location in the unit interval. Opening an outlet at the selected location is costless, yet any firm may open at most one outlet. As set forth more generally in Osborne and Pitchik (1986), the expected payoff of firm 1 when it chooses the location $z \in [0,1]$ and each of the competitors $2, \ldots, n$ randomizes according to a distribution F is given as

$$\Pi(z) = (n-1)\int_z^1 f(y)(1-F(y))^{n-2}\frac{z+y}{2}dy + \sum_{k=1}^{n-2}\binom{n-1}{k}k(n-k-1) \qquad [1]$$

$$\cdot \int_0^z \int_z^1 f(x)f(y)F(x)^{k-1}(1-F(y))^{n-k-2}\frac{y-x}{2}dydx$$

$$+(n-1)\int_0^z f(x)F(x)^{n-2}\left(1-\frac{z+x}{2}\right)dx,$$

6 Drinen, Kennedy, and Priestley (2009) study non-equilibrium aspects of the same problem, assuming that all competitors randomize independently according to a common exogenous distribution. For instance, in the uniform case with $n \geq 5$ players, the payoff function is shown to possess two small peaks, which are located close to $2/n+2$ and $n/n+2$, respectively. Moreover, expected payoffs are nearly constant between the two peaks and quickly decline to zero at the boundary of the unit interval. Thus, like in the present paper, there is an overrepresentation of (moderate) extremes. The author is grateful to the referee for bringing his attention to that interesting paper.

where $f = F'$ denotes the density of the equilibrium distribution.[7] The right-hand side of eq. [1] obviously reflects the variety of possible scenarios for the representative firm: Ending up left of all $n-1$ competing firms; then, for $k = 1, \ldots, n-2$, having a total of k competing firms to the left and $n-k-1$ competing firms to the right; or, finally, ending up right of all other firms.[8]

For a mixed-strategy equilibrium to be *doubly symmetric*, it is required that (i) all firms use the same mixed strategy F, and (ii) the strategy F is unchanged when reflected at the midpoint of the location interval, i.e., $F(1-z) = 1 - F(z)$ for all $z \in [0, 1]$. The following result summarizes what is known about the doubly symmetric mixed-strategy equilibrium of the location game in the uniform case.

Theorem 1. *For $n \geq 3$, there exists a doubly symmetric mixed-strategy equilibrium $F = F_n$, where the distribution F has support $[\alpha, 1-\alpha]$ for some $\alpha = \alpha_n \in [0, 1/2)$. For $n = 3$, the equilibrium is unique, and such that individual location choices are distributed uniformly over the interval $[1/4, 3/4]$. Moreover, if F_n is twice continuously differentiable and converges uniformly (including in terms of its first and second derivatives) to some twice continuously differentiable F_∞, then F_∞ induces a uniform distribution of location choices on the unit interval.*

Proof. See Osborne and Pitchik (1986, Prop. 3 and 4). The case $n = 3$ is treated in Shaked (1982). □

3 Discussion of the first-order condition

For any given number of competitors $k \geq 1$, consider the function

$$G_k(z) = \int_\alpha^z F(x)^k dx. \qquad [2]$$

As will become clear from the proof of the lemma below, $G_k(z)$ corresponds to the average distance between z and the highest of k lower locations.[9] Using this notation, marginal expected payoffs of the representative firm may be written in a relatively compact way.

7 Only distributions allowing a density will be considered in this paper.

8 To capture professional forecasting, suppose that a macroeconomic indicator $\xi \in \mathbb{R}$ is distributed ex-ante according to an uninformative uniform prior, and that experts have access to privileged information $\xi_0 \in \mathbb{R}$, in the sense that ξ lies somewhere in the interval $[\xi_0, \xi_0 + 1]$. The location game may then be understood as a contest, in which the forecaster whose estimate $\xi_0 + z$ turns out to be closest to the true state of the world receives a prize.

9 Similarly, provided F is symmetric, $G_k(1-z)$ corresponds to the average distance between z and the lowest of k higher estimates.

Lemma 1. *On the support of F, firm 1's marginal expected payoffs are given as*

$$\Pi'(z) = -\varphi(z) + \varphi(1-z) + f(z)\{\psi(z) - \psi(1-z)\}, \qquad [3]$$

where $\varphi(z) = F(z)^{n-1}/2$ *and*

$$\psi(z) = \frac{1}{2}\sum_{k=1}^{n-2}\binom{n-1}{k}kF(z)^{k-1}G_{n-k-1}(1-z) + (n-1)(1-z)F(z)^{n-2}. \qquad [4]$$

Proof. See the Appendix. □

Condition [3] captures two pairs of mirror-image effects resulting from a marginal shift (to the right) in firm 1's location. First, there is a marginal cost $\varphi(z)$, due to a reduced market share in the scenario in which firm 1's location is the right-most, and a mirror-image marginal benefit $\varphi(1-z)$, due to an increased market share in the scenario in which firm 1's location is the left-most. Second, there is a marginal benefit, represented by $\psi(z)$ and measured in units of the density, due to an increased probability that the locations of any given set of competitors end up left of firm 1's estimate, and a mirror-image cost represented by $\psi(1-z)$, due to a reduced probability that the locations of any complementary set of competitors end up right of firm 1's location. The doubly symmetric mixed-strategy equilibrium just balances these two pairs of effects at any point of the support interval.

Setting marginal payoffs to zero, one finds the key equation

$$f(z) = \frac{\varphi(z) - \varphi(1-z)}{\psi(z) - \psi(1-z)}. \qquad [5]$$

An obvious obstacle to interpreting eq. [5] as a differential equation in the usual meaning of the term is that the functions φ and ψ are evaluated at both z and $1-z$. This problem is addressed by a functional equation that is stated in the following lemma.

Lemma 2. *The functions* $G_1, G_2, \ldots$ *satisfy the functional equation*

$$G_k(1-z) = C_k - z - \sum_{m=1}^{k}(-1)^m\binom{k}{m}G_k(z) \qquad [6]$$

for any integer $k \geq 1$, *with constants*

$$C_k = \frac{1}{2} + \sum_{m=1}^{k-1}(-1)^m\binom{k}{m}G_m\left(\frac{1}{2}\right) + \left\{1 + (-1)^k\right\}G_k\left(\frac{1}{2}\right). \qquad [7]$$

Proof. See the Appendix. □

4 Equilibrium characterization

After these preparations, the equilibrium distribution can be characterized as the solution of a boundary value problem with a relatively simple structure.

Theorem 2. *Let $n \geq 3$. Then there exists a function $\Phi_n : \mathbb{R}^{2n-2} \to \mathbb{R} \cup \{+\infty, -\infty\}$ such that any doubly symmetric mixed-strategy equilibrium $F = F_n$ of the location game with n firms corresponds to the first element of a tuple*

$$\left(\tilde{F}, \tilde{G}_1, \ldots, \tilde{G}_{n-2}, \tilde{C}_1, \ldots, \tilde{C}_{n-2}, \tilde{\alpha}\right), \qquad [8]$$

composed of functions $\tilde{F}, \tilde{G}_1, \ldots, \tilde{G}_{n-2} : [\tilde{\alpha}, 1-\tilde{\alpha}] \to \mathbb{R}$ and constants $\tilde{C}_1, \ldots, \tilde{C}_{n-2} \in \mathbb{R}, \tilde{\alpha} \in [0, 1/2)$, such that [8] satisfies the system of ordinary first-order differential equations

$$\tilde{F}'(z) = \Phi_n(\tilde{F}(z), \tilde{G}_1(z), \ldots, \tilde{G}_{n-2}(z), \tilde{C}_1, \ldots, \tilde{C}_{n-2}, z), \qquad [9]$$

$$\tilde{G}_k'(z) = \tilde{F}(z)^k \qquad (k = 1, \ldots, n-2), \qquad [10]$$

as well as the boundary conditions $\tilde{F}(\tilde{\alpha}) = \tilde{G}_1(\tilde{\alpha}) = \ldots = \tilde{G}_{n-2}(\tilde{\alpha}) = 0$, $\tilde{F}(1/2) = 1/2$, and

$$\tilde{C}_k = \frac{1}{2} + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} \tilde{G}_m\left(\frac{1}{2}\right) + \left\{1 + (-1)^k\right\} \tilde{G}_k\left(\frac{1}{2}\right) \qquad [11]$$

for $k = 1, \ldots, n-2$. Conversely, if the first component $\tilde{F}$ of a solution of the boundary value problem stated above is restricted to be monotone increasing and symmetric with respect to a reflection at $z = 1/2$, then $\tilde{F}$ represents a doubly symmetric mixed-strategy equilibrium of the location game.

Proof. See the Appendix. □

The proof of Theorem 2 is constructive. Specifically, the function Φ_n used in the characterization simply corresponds to the right-hand side of eq. [5].

In the case $n = 3$, one can check that the two-dimensional system (9–10) reduces to the differential equation

$$\tilde{F}'(z) = \frac{2\tilde{F}(z) - 1}{4\tilde{F}(z) - 6z + 1}, \qquad [12]$$

with boundary conditions $\tilde{F}(\tilde{\alpha}) = 0$ and $\tilde{F}(1/2) = 1/2$.[10] As shown by Shaked (1982), eq. [12] has precisely one solution satisfying $\tilde{F}(1/2) = 1/2$. Thus, the unique solution of the boundary value problem is $\tilde{F}(z) = 2z - 1/2$, with $\tilde{\alpha} = 1/4$.

In cases where $n \geq 4$, the differential equation [9] becomes more involved, so that an explicit solution is not readily available. In particular, there is no obvious substitution that would simplify the equation.[11] We also checked that, in general, there is no distribution with a quadratic density function that solves eq. [9]. However, the characterization paves the way for a numerical computation of the equilibrium distribution.

5 Numerical analysis

An effective way to approximate the equilibrium is a "shooting method" that works with trajectories starting at the midpoint of the location interval.[12] For intuition, note that the starting point of the trajectory at $z = 1/2$ is an $(n-1)$-dimensional vector

$$X_0 = \left(F\left(\frac{1}{2}\right), G_1\left(\frac{1}{2}\right), \ldots, G_{n-2}\left(\frac{1}{2}\right) \right), \qquad [13]$$

whose first component is fixed through the boundary condition $F(1/2) = 1/2$, whereas the remaining components $G_1(1/2), \ldots, G_{n-2}(1/2)$ are initially unknown. Any given approximation for X_0 may then be improved by adapting the values $G_1(1/2), \ldots, G_{n-2}(1/2)$ until the corresponding trajectory satisfies the remaining boundary conditions at the boundary of the support interval with sufficient accuracy.

The details of the approximation are described below. The unknown components of the vector X_0 were initialized with the corresponding values for the uniform distribution, i.e. with

$$G_k\left(\frac{1}{2}\right) = \int_0^{1/2} z^k dz = \frac{1}{k+1}\left(\frac{1}{2}\right)^{k+1} \qquad [14]$$

10 More generally, it can be seen as a consequence of Lemma 2 that, for n odd, the function Φ_n defined in the proof of Theorem 2 does not depend on $\hat{G}_{n-2}$. Thus, for n odd, the dimension of system [9–10] reduces to $n-2$. For n even, however, this simplification is not possible.

11 E.g., in the case $n = 4$, an application of Shaked's (1982) substitution $h(z) = \left(\tilde{F}(z) - 1/2\right)/(z - 1/2)$ does not lead to a substantial simplification of the three-dimensional system [9–10].

12 The alternative computation of trajectories from the boundary of the equilibrium support proved to be numerically instable.

for $k = 1, \ldots, n-2$.[13] The iteration repeated the following steps. First, the gradient of the trajectory at the midpoint of the location interval was calculated using the relationship[14]

$$f\left(\frac{1}{2}\right) = \frac{1 + 2^{n-3}}{(n-2)\left\{1 + \sum_{k=1}^{n-3} \binom{n-3}{k} 2^k G_k\left(\frac{1}{2}\right)\right\}}. \qquad [15]$$

This equation was also employed to approximate the gradient of the trajectory in a small neighborhood of the midpoint. Next, the trajectory itself was computed on the basis of a discrete variant of system [9–10] with a grid width of $\varepsilon = 10^{-5}$.[15] Finally, the value of $1 - \alpha$ was determined to be the left-most grid point z at which the first component of the trajectory exceeded unity. The multivariate approximation was executed by a solver plug-in of a standard spreadsheet software, where we used $\sum_{k=1}^{n-2} (G_k(\alpha))^2 < 10^{-9}$ as a stopping condition.

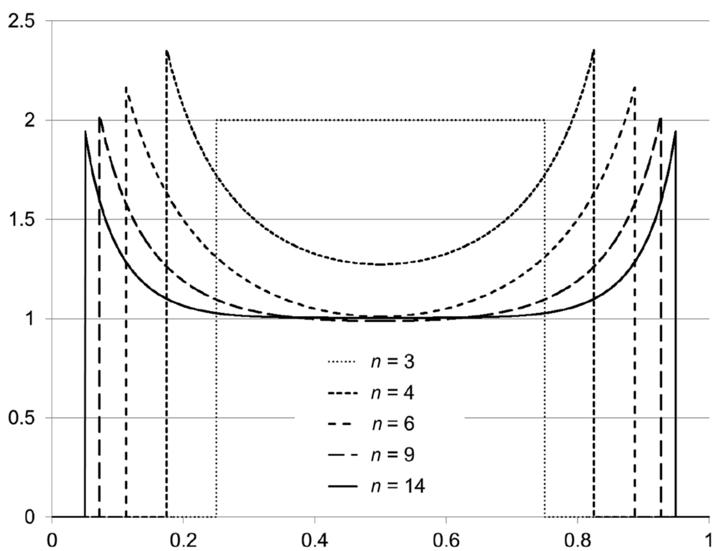

Figure 1: The density of the doubly symmetric mixed-strategy equilibrium for selected values of n

Figure 1 shows the doubly symmetric mixed-strategy equilibrium for selected values of n. As can be seen, the numerical density f is strictly M-shaped

13 Changing the initial conditions within reasonable bounds did not have any visible effect on the numerical approximations.
14 A proof of eq. [15] can be found in the Appendix.
15 A grid width of $\varepsilon = 10^{-4}$ was sufficient to obtain convergence for $n \leq 7$.

when $n \geq 4$.[16] This finding is somewhat puzzling because it implies that the equilibrium distribution for a finite number of $n \geq 4$ firms differs qualitatively from the respective uniform distributions in the cases $n = 3$ and $n \to \infty$.[17]

A second finding from the numerical analysis is that, as n increases, the locations chosen by any individual firm cover a larger support and become more dispersed (in the sense of a mean-preserving spread). Indeed, by comparing the antiderivatives of the respective distribution functions for n and $n+1$ firms, we verified within the range of considered examples that an increase in the number of firms implies a second-order stochastic dominance relationship between the equilibrium distributions. Thus, e.g., the distribution for $n = 11$ firms is a mean-preserving spread of the distribution for $n = 10$ firms. At least the widening of the equilibrium support is intuitive, however, because an increase in the population density would probably reduce expected payoffs more substantially in the interior of the support than at the boundary.

6 Concluding remarks

In this paper, we characterized the doubly symmetric mixed-strategy equilibrium in the Hotelling game of pure location for $n \geq 4$ firms and subsequently used the characterization to compute the equilibrium for small values of n. It turned out that, in all considered examples, the equilibrium overrepresents locations at the periphery of its support interval. Moreover, competition tends to expand the range of locations used in equilibrium, and to disperse the equilibrium distribution in the sense of a mean-preserving spread. These findings easily translate into testable predictions for spatial competition and political theory.[18]

As for the application to strategic forecasting, the analysis implies that forecasts will tend to be diverse (including moderately extreme) even in the absence of private information. Moreover, competition among forecasters

16 The graphs for the other calculated examples look similar. Moreover, no indication was found that the shape of the equilibrium distributions would structurally change for even larger values of n.

17 Huck, Müller, and Vriend (2002) hypothesize that the probability of getting "squeezed" between two competitors should be relatively small to make locations at the center as attractive as locations at the periphery. However, that intuition does not really explain our findings because the same intuition should apply likewise in the cases $n = 3$ and $n \to \infty$, where the equilibrium is, however, not markedly M-shaped.

18 For example, if politicians maximize their votes and if there are at least four parties, then moderately extreme platforms should be more common than central platforms. Furthermore, the existence of ultra-extreme platforms should correlate with the number of political parties.

may be counterproductive. Both observations clearly seem worthwhile to be made.[19]

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Appendix: Proofs

Proof of Lemma 1. Differentiation of eq. [1] yields

$$\begin{aligned}\Pi'(z) = {} & \frac{n-1}{2}\int_z^1 f(y)(1-F(y))^{n-2}dy - (n-1)f(z)(1-F(z))^{n-2}z \\ & + f(z)\sum_{k=1}^{n-2}\binom{n-1}{k}k(n-k-1) \\ & \cdot\left\{F(z)^{k-1}\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy\right. \\ & \left. -(1-F(z))^{n-k-2}\int_0^z f(x)F(x)^{k-1}\frac{z-x}{2}dx\right\} \\ & + (n-1)f(z)F(z)^{n-2}(1-z) - \frac{n-1}{2}\int_0^z f(x)F(x)^{n-2}dx.\end{aligned} \tag{16}$$

We will now rewrite the two integrals in the interior of the curly brackets. First, applying integration by parts, one can check that

$$\int_0^z f(x)F(x)^{k-1}\frac{z-x}{2}dx = \frac{F(x)^k}{k}\frac{z-x}{2}\bigg|_{x=0}^{x=z} + \frac{1}{2}\int_0^z \frac{F(x)^k}{k}dx \tag{17}$$

$$= \frac{G_k(z)}{2k}, \tag{18}$$

19 Competition at the level of individual estimates is indeed sometimes avoided. This has been the case, for example, for the Joint Economic Forecast that has been prepared twice yearly since 1950 by leading economic research institutes on behalf of the German Ministry of Economic Affairs. For a general approach to mitigating the inefficiencies caused by strategic information transmission, see Ewerhart and Schmitz (2000).

where we have used that $F(x) = 0$ for $x \in [0, \alpha]$. Second, applying the substitution $x = 1 - y$, and noting the symmetry property $1 - F(1-x) = F(x)$, one obtains

$$\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy = \int_0^{1-z} f(x)F(x)^{n-k-2}\frac{1-z-x}{2}dx. \tag{19}$$

Hence, eqs. [17–18], with z and k replaced by $1-z$ and $n-k-1$, respectively, imply

$$\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy = \frac{G_{n-k-1}(1-z)}{2(n-k-1)}. \tag{20}$$

Next, the terms obtained for the integrals via eqs. [17–18] and [20] are plugged into eq. [16]. Using also the obvious relationships

$$\frac{n-1}{2}\int_z^1 f(y)(1-F(y))^{n-2}dy = \frac{1}{2}(1-F(z))^{n-1}, \tag{21}$$

$$\frac{n-1}{2}\int_0^z f(x)F(x)^{n-2}dx = \frac{1}{2}F(z)^{n-1}, \tag{22}$$

one arrives at

$$\begin{aligned}\Pi'(z) = {} & \frac{1}{2}(1-F(z))^{n-1} - (n-1)f(z)(1-F(z))^{n-2}z + \frac{f(z)}{2}\sum_{k=1}^{n-2}\binom{n-1}{k} \\ & \cdot \left\{kF(z)^{k-1}G_{n-k-1}(1-z) - (n-k-1)(1-F(z))^{n-k-2}G_k(z)\right\} \\ & + (n-1)f(z)F(z)^{n-2}(1-z) - \frac{1}{2}F(z)^{n-1}.\end{aligned} \tag{23}$$

A simple re-ordering of terms, mapping index k to $n-k-1$ and vice versa, finally shows that

$$\begin{aligned}&\sum_{k=1}^{n-2}\binom{n-1}{k}(n-k-1)(1-F(z))^{n-k-2}G_k(z) \\ &= \sum_{k=1}^{n-2}\binom{n-1}{k}k(1-F(z))^{k-1}G_{n-k-1}(z).\end{aligned} \tag{24}$$

Using now eq. [24] to rewrite eq. [23], and exploiting the symmetry of F once more, the lemma follows. □

Proof of Lemma 2. By definition, $G_k(1-z) = \int_\alpha^{1-z} F(x)^k dx$. Splitting the integral and subsequently exploiting symmetry, one finds

$$G_k(1-z) = \int_\alpha^{1-\alpha} F(x)^k dx - \int_{1-z}^{1-\alpha} F(y)^k dy \quad [25]$$

$$= G_k(1-\alpha) - \int_\alpha^z F(1-x)^k dx \quad [26]$$

$$= G_k(1-\alpha) - \int_\alpha^z (1-F(x))^k dx. \quad [27]$$

Thus,

$$G_k(1-z) = G_k(1-\alpha) - z + \alpha - \sum_{m=1}^{k} (-1)^m \binom{k}{m} G_m(z), \quad [28]$$

for any $z \in [\alpha, 1-\alpha]$. Evaluating eq. [28] at $z = 1/2$ yields

$$G_k(1-\alpha) = \frac{1}{2} - \alpha + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} G_m\left(\frac{1}{2}\right) + \left\{1 + (-1)^k\right\} G_k\left(\frac{1}{2}\right). \quad [29]$$

Plugging this back into eq. [28] proves the claim. □

Proof of Theorem 2. *(Necessity)* To construct Φ_n, one first writes differential equation [5] in explicit form, i.e., using the definitions of $\varphi(z)$ and $\psi(z)$ provided in Lemma 1. This yields

$$F'(z) = \left\{\frac{F(z)^{n-1} - F(1-z)^{n-1}}{2}\right\} \quad [30]$$

$$\cdot \left\{ \begin{array}{l} (n-1)(1-z)F(z)^{n-2} - (n-1)zF(1-z)^{n-2} + \frac{1}{2}\sum_{k=1}^{n-2} \binom{n-1}{k} k \\ \cdot \left\{F(z)^{k-1} G_{n-k-1}(1-z) - F(1-z)^{k-1} G_{n-k-1}(z)\right\} \end{array} \right\}^{-1}.$$

Re-ordering the terms of the sum by mapping index k to $n-k-1$, and subsequently using the relationship $\binom{n-1}{k}(n-k-1) = (n-1)\binom{n-2}{k}$, one obtains

$$F'(z) = \left\{\frac{F(z)^{n-1} - F(1-z)^{n-1}}{n-1}\right\} \quad [31]$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)F(z)^{n-2} - 2zF(1-z)^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{ F(z)^{n-k-2} G_k(1-z) - F(1-z)^{n-k-2} G_k(z) \right\} \end{array} \right\}^{-1}.$$

Replacing all occurrences of $F(1-z)$ by $1-F(z)$, and similarly, all occurrences of $G_1(1-z), \ldots, G_{n-2}(1-z)$ by the corresponding expressions in Lemma 2, we arrive at

$$F'(z) = \left\{ \frac{F(z)^{n-1} - (1-F(z))^{n-1}}{n-1} \right\} \tag{32}$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)F(z)^{n-2} - 2z(1-F(z))^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{ F(z)^{n-k-2} \left\{ C_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} G_k(z) \right\} \right. \\ \qquad \left. - (1-F(z))^{n-k-2} G_k(z) \right\} \end{array} \right\}^{-1}.$$

In analogy with eq. [32], define the function $\Phi_n : \mathbb{R}^{2n-2} \to \mathbb{R} \cup \{+\infty, -\infty\}$ by

$$\Phi_n(\hat{F}, \hat{G}_1, \ldots, \hat{G}_{n-2}, \hat{C}_1, \ldots, \hat{C}_{n-2}, z) = \left\{ \frac{\hat{F}^{n-1} - (1-\hat{F})^{n-1}}{n-1} \right\} \tag{33}$$

$$\cdot \left\{ \begin{array}{c} 2(1-z)\hat{F}^{n-2} - 2z(1-\hat{F})^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{ \hat{F}^{n-k-2} \left\{ \hat{C}_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} \hat{G}_m \right\} - (1-\hat{F})^{n-k-2} \hat{G}_k \right\} \end{array} \right\}^{-1}.$$

Then, by construction, F is the first component of a solution of the boundary value problem stated in Theorem 2, thereby proving the first part of the theorem.

(Sufficiency) Suppose that $\tilde{F}$ is monotone increasing, and that $\tilde{F}$ is symmetric in the sense that $\tilde{F}(1-z) = 1 - \tilde{F}(z)$ for any $z \in [\tilde{\alpha}, 1-\tilde{\alpha}]$. Then, from $\tilde{G}_k(\tilde{\alpha}) = 0$ and $\tilde{G}'_k(z) = \tilde{F}(z)^k$, it follows that $\tilde{G}_k(z) = \int_{\alpha}^{z} \tilde{F}(x)^k dx$. From the symmetry of $\tilde{F}$, one may derive just as in the proof of Lemma 2 that

$$\tilde{G}_k(1-z) = \tilde{C}_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} \tilde{G}_k(z), \tag{34}$$

for any integer $k \geq 1$, where

$$\tilde{C}_k = \frac{1}{2} + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} \tilde{G}_m\left(\frac{1}{2}\right) + \left\{1 + (-1)^k\right\} \tilde{G}_k\left(\frac{1}{2}\right). \qquad [35]$$

By assumption, eq. [32] holds with $F, G_1, \ldots, G_{n-2}, C_1, \ldots, C_{n-2}$ replaced by $\tilde{F}, \tilde{G}_1, \ldots, \tilde{G}_{n-2}, \tilde{C}_1, \ldots, \tilde{C}_{n-2}$. Using the symmetry of $\tilde{F}$ and functional equation [34] for $k = 1, \ldots, n-2$, one arrives at

$$\begin{aligned}\tilde{F}'(z) = &\left\{\frac{\tilde{F}(z)^{n-1} - \tilde{F}(1-z)^{n-1}}{n-1}\right\} \\ &\cdot \left\{\begin{array}{l} 2(1-z)\tilde{F}(z)^{n-2} - 2z\tilde{F}(1-z)^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{\tilde{F}(z)^{n-k-2}\tilde{G}_k(1-z) - \tilde{F}(1-z)^{n-k-2}\tilde{G}_k(z)\right\} \end{array}\right\}^{-1}.\end{aligned} \qquad [36]$$

Hence, invoking Lemma 1, $\tilde{F}$ solves the first-order condition, and expected payoffs are constant on the interval $[\tilde{\alpha}, 1-\tilde{\alpha}]$. Moreover, by the nature of expected payoffs in the location game, any location $z < \tilde{\alpha}$ yields strictly lower expected payoffs than $\tilde{\alpha}$, and similarly, any location $z > 1-\tilde{\alpha}$ yields strictly lower expected payoffs than $1-\tilde{\alpha}$. Thus, $\tilde{F}$ really corresponds to a doubly symmetric mixed-strategy equilibrium. □

Proof of eq. [15]. A straightforward application of the rule of L'Hôpital to differential equation [5] shows that

$$f\left(\frac{1}{2}\right) = \frac{\varphi'\left(\frac{1}{2}\right)}{\psi'\left(\frac{1}{2}\right)}. \qquad [37]$$

Noting that $F(1/2) = 1/2$, one readily verifies that

$$\varphi'\left(\frac{1}{2}\right) = \frac{n-1}{2^{n-1}} f\left(\frac{1}{2}\right). \qquad [38]$$

Moreover, using $G'_{n-k-1}(1/2) = F^{n-k-1}(1/2) = 1/2^{n-k-1}$, one can check that

$$\begin{aligned}\psi'\left(\frac{1}{2}\right) = &f\left(\frac{1}{2}\right) \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k(k-1)}{2^{k-1}} G_{n-k-1}\left(\frac{1}{2}\right) - \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{2^{n-1}} \\ &- \frac{n-1}{2^{n-2}} + f\left(\frac{1}{2}\right) \frac{(n-1)(n-2)}{2^{n-2}}.\end{aligned} \qquad [39]$$

Exploiting the identities

$$\sum_{k=1}^{n-2}\binom{n-1}{k}\frac{k(k-1)}{2^{k-1}}G_{n-k-1}\left(\frac{1}{2}\right)$$

$$=\frac{1}{2^{n-2}}\sum_{k=1}^{n-2}\binom{n-1}{k}k(k-1)2^{n-k-1}G_{n-k-1}\left(\frac{1}{2}\right) \quad [40]$$

$$=\frac{1}{2^{n-2}}\sum_{k=1}^{n-2}\binom{n-1}{k}(n-k-1)(n-k-2)2^{k}G_{k}\left(\frac{1}{2}\right) \quad [41]$$

$$=\frac{(n-1)(n-2)}{2^{n-2}}\sum_{k=1}^{n-3}\binom{n-3}{k}2^{k}G_{k}\left(\frac{1}{2}\right) \quad [42]$$

and

$$\sum_{k=1}^{n-2}\binom{n-1}{k}k=(n-1)\sum_{k=1}^{n-2}\binom{n-2}{k-1}=(n-1)(2^{n-2}-1), \quad [43]$$

it follows that

$$\psi'\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)\frac{(n-1)(n-2)}{2^{n-2}}\left\{1+\sum_{k=1}^{n-3}\binom{n-3}{k}2^{k}G_{k}\left(\frac{1}{2}\right)\right\} \quad [44]$$

$$-\frac{(n-1)(1+2^{n-2})}{2^{n-1}}.$$

Plugging now eqs. [38] and [44] into eq. [37], and subsequently solving for $f(1/2)$, one arrives at eq. [15]. □

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